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On Locally b-Closed, b-Pre-Open & sb-Generalized Closed Sets In Topological Spaces

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Abstract:
The present paper contains a discourse on locally b-closed, b-pre-open and s b-generalized closed sets and their connectivity with regular open sets. Also, the equivalence of an open set with a b-open as well as a D(c,b) set is obtained as a part of the paper. The basic properties of these classes of generalized sets have been found a place in this paper.

Keywords: b-open, b-pre-open, locally b-closed, b-semi-open, sb-generalized closed set, t-set, b-t-set, B-set, b-B-set & D(c, b) set.

1. Introduction & Preliminaries:-

The mathematical papers [1,2], produced by D. Andrijevic, introduce and investigate semi-pre-open and b-open sets which are weak forms of open sets. Levine [3] started the study of generalized open sets with the introduction of semi-open sets & Njasted [4] studied α-open sets; Mashour et.al. [5] introduced pre-open sets. At the same time Bourbaki [6] invented the concept of locally closed sets.
Obviously, one of the most significant concepts in topology is the notion of b-open sets which was discussed by E. Ekici & M. Caldas [7] under the name γ-open set. The notions mentioned in [3], [4] & [5] were defined using the closure operator (cl) and the interior operator (int) in the following manner:

Definition (1.1) :
A subset A of a space (X,T) is called
(a) a semi-open [3] set if $A \subset cl(int(A))$ and semi closed set if $int(cl(A)) \subset A$.
(b) a pre-open [5] set if $A \subset int(cl(A))$ and pre-closed set if $cl(int(A)) \subset A$.
(c) an α-open [4] set if $A \subset int(cl(int(A)))$ and α -closed set if $cl(int(cl(A))) \subset A$.
The semi-pre-open set, called by D.Andrijevic [1], was introduced under the name β-open set by M.E. Abd. El-Moneef et.al [7] as:

Definition (1.2):
A subset A of a space (X,T) is called a semi-pre-open [1] or β-open [7] set if $A \subset cl(int(cl(A)))$ and a semi-pre closed or β-closed set if $int(cl(int(A))) \subset A$
Now, the new class of generalized open sets given by D. Andrijevic under the name of b-open sets is as :

Definition (1.3):
A subset A of a space (X,T) is called a b-open set [2] if $A \subset cl(int(A)) \cup int(cl(A))$ and a b-closed set [8] if $cl(int(A)) \cap int(cl(A)) \subset A$.
All the definition (1.1), (1.2) & (1.3) are different and independent [8].
The classes of pre-open, semi-open, α-open, semi-pre-open and b-open subsets of a space (X,T) are usually denoted by PO(X,T), SO(X,T), T^{α} , SPO(X,T) and BO(X,T) respectively. All of them are larger than T and closed under forming arbitrary unions.
In 1996, D. Anderjevic. made the fundamental observation:

Prop. : (1.4): For every space (X, T)
$PO(X, T) \cup SO(X, T) \subseteq BO(X, T) \subseteq SPO(X,T)$ holds but none of these implications can be reversed [8]
:
Prop. (1. 5): characterization [8]:
(a) S is semi-preopen iff $S \subseteq Sint(Scl\ S)$
(b) S is semi-open iff $S \subseteq Scl(Sint\ S)$
(c) S is pre-open iff $S \subseteq pint(pcl\ S)$
(d) S is b-open iff $S \subseteq pcl(pint\ S)$ where $S \subseteq X$ and (X,T) is a space.
Next, J.Tong [9] introduced the concept of t-set and B-set in a topological space as :

Definition (1.6):
A subset A of a space (X, T) is called
a) a t-set [9] if $int(A) = int(cl(A))$
b) a B-set [9] if $A = U \cap V$ where $U \in T$ & V is a t-set
c) Locally closed [6] if $A = U \cap V$ where $U \in T$ & V is a closed set
d) Locally b-closed [10] if $A = U \cap V$ where $U \in T$ & V is a b-closed set
All through this paper (X,T) and (Y□,σ) stand for topological spaces with no separation axioms assumed unless otherwise stated.
The intersection of all b-closed sets of X containing A is known to be the bcl (A) of A. The union of all b-open sets of X contained in A is known to be b-interior of A and is denoted by bint(A).

2. Locally b-closed & D(c, b) sets :-
This section deals with the theorems related to locally b-closed sets and the concept of a D(c,b) set.
Theorem (2.1): If A is a subset of an extremely disconnected space (X, T), then the following statements are equivalent:
a) A is open
b) A is b-open & locally closed.

Proof: (a) $\Rightarrow$ (b):
Since, every open set is a b-open set; hence A is b-open. Also,

$A \cap X = A$, where $A \in$ □ T & X is closed, gives that A is locally closed.

(b) ⇒ (a):

Let A be b-open & locally closed.

Then, $A \subseteq int(cl(A)) \cup cl(int(A))$ & $A = U \cap cl(A)$ where $U \in T$. [6]

Now, $A \subseteq U \cap [int(cl(A) \cup cl(int(A))]$

$= U \cup int(cl(A))$ [since, $cl(int(A)) \subseteq int(cl(A))$ for an extremally disconnected space]

$= (intU) \cap int(cl(A))$

$= int(U \cap cl(A))$

$= int A$

i.e $A \subseteq int(A)$, so that A is open.

Hence the theorem

Definition (2.2): A subset A of a space (X, T) is called D(c,b)-set if int(A) = bint(A).The concepts of b-open and D(c,b) sets are independent as illustrated by the following example:

Example:

Let $X = \{a,b,c\}$, $T = \{\phi,\{a\},\{a,c\},X\}$, then the simple computation gives that

$T_b = BO(X, T) = \{\phi,\{a\},\{a,b\},\{a,c\},X\}$

And the class of all D(c,b)-sets = $\{\phi,\{a\},\{b\},\{c\},\{a,c\},\{b,c\}\}$

Since, {a,b} is b-open but not D(c,b)-set and {b,c} is a D{c,b}-set but not a b-open set.

Hence, b-open set & D(c,b)-set are independent.

Theorem (2.3): If $A \subseteq X$ where (X,T) is a space, then [A is open] ⇔ [A is a b-open set and a D(c,b)-set]

Proof:-

Let A be an open set in the space (X,T), then A is also b-open.

Again, A = int(A) and A=bint(A), So int(A)=bint(A)

Hence, A is a D(c,b) set

Conversely, let A be a b-open set as well as a D(c,b) set. So, A=bint(A) & int(A)=bint(A).

Hence, A=int(A) which means that A is open.

Theorem (2.4): If H be a subset of (X,T), then H is locally b-closed iff there exists an open set $U \subseteq X$ in the manner that $H=U \cap bcl(H)$.

Proof: Assume that H is a subset of X in a space (X,T).

Let H be locally b-closed.

Then $H = U \cap F$ where U is open & F is b-closed.

This means that $H \subseteq U$ and $H \subseteq F$.

Again, $H \subseteq bcl(H) \subseteq bcl(F)=F$.

Hence, $H \subseteq U \cap bcl(H)$.

& Also, $U \cap bcl(H) \subseteq U \cap bcl(F)$

$= U \cap F$

$= H$

Therefore, $H = U \cap bcl(H)$.

Conversely, assume that $H=U \cap bcl(H)$ where $U \in T$ in the space (X,T).

Since, bcl(H) is b-closed, hence $H=U \cap bcl(H)=U \cap V$ where $U \in T$ &

$V=bcl(H) \in Bc(X,T)$

This concludes that H is a locally b-closed set.

Theorem (2.5): If A be a locally b-closed subset of a space (X, T),

Then (a) bcl (A)-A is a b-closed set (b) $[A \cup (X-bcl(A))]$ is b-open & (c) $A \subseteq$ bint$[A \cup \{X-bcl(A)\}]$.

Proof:

Let A be a locally b-closed subset of a space (X, T).

(a) Using theorem (2.4), there exists an open set U in X such that

$A = U \cap bcl(A)$.

Now, $bcl(A) - A = bcl(A) - [U \cap bcl(A)]$

$= bcl(A) \cap [U \cap bcl(A)]^C$

$= bcl(A) \cap [X-\{U \cap bcl(A)\}]$

$= bcl(A) \cap [(X-U) \cup (X-bcl(A)]$

$= [bcl(A) \cap (X-U)] \cup [bcl(A) \cup (X-bcl(A))]$

$= [bcl(A) \cap (X-U)] \cup \phi$

$= bcl(A) \cap (X-U)$

$= bcl(A) \cap U^C$ = Intersection of two b-closed sets.

[As U^C is closed & so b-closed]

= A b-closed set

Hence, bcl(A)-A is b-closed.

(b) Since, bcl(A)- A is b-closed, hence [X-(bcl(A) - A)] is b-open.

Now, $X-\{bcl(A) - A\} = X-\{bcl(A) \cap A^C\}$

$= \{X-bcl(A)\} \cup (X-A^C)$

$= \{X-bcl(A)\} \cup A$

i.e, $X-\{bcl(A) - A\} = A \cup (X-bcl(A))$

$\Rightarrow [A \cup (X-bcl(A)]$ is b-open.

(c) Naturally, $A \subseteq A \cup (X-bcl(A)) =$ bint$[A \cup (X-bcl(A)]$

Corollary: The intersection of a locally b-closed set and a locally closed set is locally b-closed.

Theorem (2.6): If (X, T) is closed under finite union of b-closed sets and A & B are separated locally b-closed sets, then $A \cup B$ is locally b-closed.

Proof:

Since, A and B are locally b-closed, hence $A = G \cap bcl(A)$ and $B= H \cap bcl(B)$ where $G, H \in T$ and (X,T) is a space.

Let, $U = G \cap (X\text{-}cl(B))$ and $V = H \cap (X\text{-}cl(A))$.

Then,
$$\begin{aligned} U \cap bcl(A) &= [G \cap (X\text{-}cl(B))] \cap bcl(A) \\ &= G \cap [(X\text{-}cl(B)) \cap bcl(A)] \\ &= G \cap [bcl(A) \cap (X\text{-}cl(B))] \\ &= A \cap [cl(B)]^C = A \end{aligned}$$

[Since, $A \cap cl(B)= \phi$ because A & B are seperated]

Similarly, $V \cap bcl(B) = B$

Next,
$$\begin{aligned} U \cap bcl(B) &\subseteq U \cap cl(B) \\ &= [G \cap (X\text{-}cl(B))] \cap cl(B) \\ &= G \cap \phi = \phi \end{aligned}$$

Similar, $V \cap bcl(A) = \phi$

Since, U and V are open sets, hence, $U \cup V$ is also an open set.

Now,
$$\begin{aligned} (U \cup V) \cap bcl\,(A \cup B) &= (U \cup V) \cap (bcl\,A \cup bcl\,B) \\ &= (U \cap bclA) \cup (V \cap bclA) \cup (U \cap bclB) \cup (V \cap bcl\,B) \\ &= A \cup \phi \cup \phi \cup B = A \cup B \end{aligned}$$

[Using, the previous results]

Consequently, $A \cup B$ is a locally b-closed set.

Hence, the theorem

3. b-semi-open, b-preopen & sb-generalized closed sets :

This section is the study of the nature of b-semi-open, b-pre-open & sb-generalized closed sets.

Definition (3.1):

A subset A of a space (X,T) is said to be

(a) a b-semi-open set if $A \subseteq cl(bint(A)$ and a b-semi-closed set if $int(bcl(A)) \subseteq A$.

(b) a b-pre-open set if $A \subseteq int(bcl(A))$ and a b-pre-closed set if $cl(bint(A)) \subseteq A$.

(c) a sb-generalized closed set if $sbcl(A) \subseteq U$ whenever $A \subseteq U$ and U is a b-pre-open set. Here sbcl(A) is the intersection of all b-semi-closure sets containing A.

Example:

Let X={a,b,c,d}, T={ϕ,X,{c},{d},{c,d},{a,c,d},{b,c,d}}

and T^C= {ϕ,X,{a},{b},{a,b},{a,b,c},{a,b,d}}

The collection of all sb-generalized closed sets = P(X) - {{c,d},{a,c,d},{b,c,d}}.

The collection of all sb-generalized open sets = P(X) - {{a},{b},{a,b}}.

Also, BPO(X,T) = T, BO(X,T) = P(X) - {{a},{b},{a,b}} = SBGO(X,T)

Hence, b-pre-open & sb-generalized open sets are independent

Lemma (3.2): If A is an open set of (X,T), then bcl(A) = scl(A) and
int (bcl(A)) = int(cl(A))

Proof:

Let A be an open set then A = int(A)

Now, $bcl(A) = scl(A) \cap pcl(A)$
$= [A \cup int(cl(A))] \cap [A \cup cl(int(A))]$
$= A \cup [int(cl(A)) \cap cl(int(A))]$
$= A \cup [int(cl(A)) \cap cl(A)]$
$= A \cup int(cl(A))$ [Since, $int(cl(A)) \subset cl(A)$]
$= scl(A)$

Again, $int(bcl(A)) = int(scl(A)) = int(cl(A))$
[since, $int(scl(A)) = int(cl(A))$]

Corollary: For any open subset A of (X,T),
$[A \subseteq int(bcl(A))] \Leftrightarrow [A \subseteq int(cl(A))]$
i.e. A is b-preopen $\Leftrightarrow$ A is pre-open.

Theorem (3.3): For a subset A of a space (X,T), the following statements are Equivalent:

(a) A is regular open
(b) A = int (bcl(A))
(c) A is b-preopen and b-t-set.

Proof: (a) $\Rightarrow$ (b):

Let A be regular open then A = int(cl(A))

By lemma (3.2), int(bcl(A)) = int(cl(A)),
i.e, int(bcl(A)) = A

(b) $\Rightarrow$ (c):

Given A = int(bcl(A))
$\Rightarrow$ $A \subseteq int(bcl(A))$ & $int(bcl(A)) \subseteq A$.
$\Rightarrow$ A is b-preopen.

Next,

A = int(bcl(A))
$\Rightarrow$ int(A) = int {int(bcl(A))}
$\Rightarrow$ int (A) = int(bcl(A))
$\Rightarrow$ A is b-t-set.

(c) $\Rightarrow$ (a):

Let A be b-preopen as well as b-t-set.
Hence, $A \subseteq int(bcl(A))$ & int(A) = int(bcl(A))
Then, combining these two, we get
$A \subseteq int(A) \subseteq A$
$\Rightarrow$ A = int(A) i.e. A is open & so A is regular open.

Hence, the theorem

Theorem (3.4): If A $\subseteq$X where (X, T) is a space,

Then [A is regular open] $\Leftrightarrow$ [A is b-preopen & sb-generalized closed]

Proof:

Let A $\subseteq$ X where (X,T) is a space such that A is a regular open set. Then, by theorem (3.3), A is a b-pre-open set.

This means that A $\subseteq$ int(bcl(A)).

Now, sbcl(A) = A $\cup$ (int(bcl(A)).
= int(bcl(A))
= int(cl(A)) [lemma(3.2)]
= A [Since, A is regular open]

$\Rightarrow$ A is sb-generalized closed.

Conversely, let A be b-pre-open as well as sb-generalized closed. i.e. sbcl(A)=A

Since, A is b-pre-open as well as sb-generalized and so A $\subseteq$ A$\Rightarrow$sbcl(A) $\subseteq$ A

Whenever A $\subseteq$ A. Thus sbcl(A)$\subseteq$ A. But A$\subseteq$sbcl(A)cousequently, A = sbcl(A) which shows that A is b-semi-closed.

Therefore, int(bcl(A)) $\subseteq$ A. Again, A, being b-pre-open, gives that A $\subseteq$ int(bcl(A)). Combining these facts, A = int(bcl(A)).

Next, A = int(bcl(A)) = int(cl(A)) [lemma (3.2)]

Consequently, A is a regular open set.

Hence, the theorem

Conclusion:

The basic important properties of locally b-closed, b-pre-open and sb-generalized closed sets have been obtained and analyzed. The future scope of study is to obtain results for decomposition of continuity via locally b-closed sets, b-pre-open as well as b-semi-open sets and sb-generalized closed sets in topological spaces.

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